

Van der Waerden Conjecture for Mixed Discriminants

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Abstract

We prove that the mixed discriminant of doubly stochastic n -tuples of semidefinite hermitian $n \times n$ matrices is bounded below by $\frac{n!}{n^n}$ and that this bound is uniquely attained at the n -tuple $(\frac{1}{n}I, \dots, \frac{1}{n}I)$. This result settles a conjecture posed by R. Bapat in 1989. We consider various generalizations and applications of this result.

1 Introduction

An $n \times n$ matrix A is called doubly stochastic if it is nonnegative entry-wise and its every column and row sum to one. The set of $n \times n$ doubly stochastic matrices is denoted by Ω_n .

Let S_n be the symmetric group, i.e. the group of all permutations of the set $\{1, 2, \dots, n\}$. Recall that the permanent of a square matrix A is defined by

$$\text{per}(A) = \sum_{\sigma \in S_n} \prod_{i=1}^n A(i, \sigma(i)).$$

The famous Van der Waerden Conjecture [17] states that

$$\min_{A \in \Omega_n} D(A) = \frac{n!}{n^n}$$

and the minimum is attained uniquely at the matrix J_n in which every entry equals $\frac{1}{n}$. The “modern” attack at the conjecture began in fifties (of 20th century), see [17] for some history, and culminated with three papers [6], [12], [14]. In a very technical paper [6] S. Friedland got very close to the desired lower bound by proving that $\min_{A \in \Omega_n} D(A) \geq e^{-n}$.

In [12] D.I. Falikman proved the lower bound $\frac{n!}{n^n}$ via ingenious and custom-made arguments. Finally, the full conjecture was proved by G.P. Egorychev in [14]. The paper [14] capitalized on the simple, but crucial, observation that the permanent is a particular case of the mixed volume or the mixed discriminant, which we will define below. Having in mind this connection, the main inequality in [12] is just a particular case of the famous Alexandrov-Fenchel inequalities [9].

Let us consider an n -tuple $\mathbf{A} = (A_1, A_2, \dots, A_n)$, where $A_i = (A_i(k, l) : 1 \leq k, l \leq n)$ is a complex $n \times n$ matrix ($1 \leq i \leq n$). Then $\det(\sum t_i A_i)$ is a homogeneous polynomial of degree n in t_1, t_2, \dots, t_n . The number

$$D(\mathbf{A}) := D(A_1, A_2, \dots, A_n) = \frac{\partial^n}{\partial t_1 \dots \partial t_n} \det(t_1 A_1 + \dots + t_n A_n) \quad (1)$$

is called the mixed discriminant of A_1, A_2, \dots, A_n .

Mixed discriminants were introduced by A.D. Alexandrov as a tool to derive mixed volumes of convex sets ([9], [10]). They are also a 3-dimensional case of multidimensional Pascal's determinants [16].

There exist many alternative ways to define mixed discriminants. Let S_n be the symmetric group, i.e. the group of all permutations of the set $\{1, 2, \dots, n\}$. Then the following identities hold.

$$D(A_1, \dots, A_n) = \sum_{\sigma, \tau \in S_n} (-1)^{sgn(\sigma\tau)} \prod_{i=1}^n A_i(\sigma(i), \tau(i)). \quad (2)$$

$$D(A_1, \dots, A_n) = \sum_{\sigma \in S_n} \det(A_\sigma), \quad (3)$$

where the i th column of A_σ is the i th column of $A_{\sigma(i)}$.

$$D(A_1, \dots, A_n) = \sum_{\sigma \in S} (-1)^{sgn(\sigma)} per(B_\sigma), \quad (4)$$

where $B_\sigma(k, l) = A_l(k, \sigma(k))$.

$$M(A_1, \dots, A_N) = \langle (A_1 \otimes \dots \otimes A_N)V, V \rangle \quad (5)$$

where the N^N -dimensional vector $V = V(i_1, i_2, \dots, i_n) : 1 \leq i_k \leq N, 1 \leq k \leq N$ is defined as follows:

$V(i_1, i_2, \dots, i_n)$ is equal to $(-1)^{sign(\tau)}$ if there exists a permutation $\tau \in S_N$, and equal to zero otherwise.

It follows from the definition of the permanent that $per(A) = D(A_1, \dots, A_n)$, where $A_j = \text{Diag}(A(i, j) : 1 \leq i \leq n), 1 \leq j \leq n$.

In a 1989 paper [3] R.B. Bapat defined the set D_n of doubly stochastic n -tuples. An n -tuple $\mathbf{A} = (A_1, \dots, A_n)$ belongs to D_n iff the following properties hold:

1. $A_i \succeq 0$, i.e. A_i is a positive semi-definite matrix, $1 \leq i \leq n$.
2. $tr A_i = 1$ for $1 \leq i \leq n$.
3. $\sum_{i=1}^n A_i = I$, where I , as usual, stands for the identity matrix.

One of the problems posed in [3] is to determine the minimum of mixed discriminants of doubly stochastic tuples

$$\min_{A \in D_n} D(A) = ?$$

Quite naturally, Bapat conjectured that

$$\min_{A \in D_n} D(A) = \frac{n!}{n^n}$$

and that it is attained uniquely at $\mathbf{J}_n =: (\frac{1}{n}I, \dots, \frac{1}{n}I)$.

In [3] this conjecture was formulated for real matrices. We will prove it in this paper for the complex case, i.e. when matrices A_i above are complex positive semidefinite and, thus, hermitian. (Recall that a square complex $n \times n$ matrix $A = \{A(i, j) : 1 \leq i, j \leq n\}$ is called hermitian if $A = A^* = \{\overline{A(j, i)} : 1 \leq i, j \leq n\}$. A square complex $n \times n$ matrix A is hermitian iff $\langle Ax, x \rangle = \langle x, Ax \rangle$ for all $x \in C^n$.) One of the main tools we will use below are necessary conditions for a local minimum under semidefinite constraints. It is very important, in this optimizational context, that the set of $n \times n$ hermitian matrices can be viewed as an n^2 -dimensional real linear space with (real) inner product $\langle A, B \rangle =: \text{tr}(AB)$.

The rest of the paper will provide a proof of Bapat's conjecture. (The lower bound $\frac{n!}{n^n}$ for real symmetric doubly stochastic n -tuples was proved in [4].)

2 Basic Facts about Mixed Discriminants

Fact 1.

$$D(X\alpha_1 A_1 Y, \dots, X\alpha_i A_i Y, \dots, X\alpha_n A_n Y) = \det(X) \cdot \det(Y) \cdot \prod_{i=1}^n \alpha_i \cdot D(A_1, \dots, A_n). \quad (6)$$

Fact 2.

$$D(x_i y_i^*, \dots, x_n y_n^*) = \det\left(\sum_{i=1}^n x_i y_i^*\right). \quad (7)$$

Here $x_i y_i^*$ is an $n \times n$ complex matrix of rank one, x_i and y_i are $n \times 1$ matrices (column-vectors), y^* is an adjoint matrix, i.e. $y^* = \overline{y^T}$.

Fact 3.

$$\begin{aligned} D(A_1, \dots, A_{i-1}, \alpha A + \beta B, A_{i+1}, \dots, A_n) = \\ \alpha D(A_1, \dots, A_{i-1}, A, A_{i+1}, \dots, A_n) + \beta D(A_1, \dots, A_{i-1}, B, A_{i+1}, \dots, A_n). \end{aligned} \quad (8)$$

Fact 4.

$D(A_1, \dots, A_n) \geq 0$ if $A_i \succeq 0$ (positive semidefinite), $1 \leq i \leq n$.

This inequality follows, for instance, from the tensor product representation (5).

Fact 5.

Suppose that $A_i \succeq 0$, $1 \leq i \leq n$. Then $D(A_1, \dots, A_n) > 0$ iff for any $1 \leq i_1 < i_2 < \dots < i_k \leq n$ the following inequality holds: $\text{Rank}(\sum_{j=1}^k A_{i_j}) \geq k$ [15].

This fact is a rather direct corollary of the Rado theorem on the rank of intersection of a matroid

of transversals and a geometric matroid, which is a particular case of the famous Edmonds' theorem on the rank of inrersection of two matroids [13].

Fact 6.

$D(A_1, \dots, A_n) > 0$ if the n -tuple (A_1, \dots, A_n) is a doubly stochastic. This fact follows from Fact 5.

Fact 7.

$$D(A_1, \dots, A_{i-1}, X, A_{i+1}, \dots, A_n) = \text{tr}(X \cdot Q_i). \quad (9)$$

where the matrix $Q_i = (\frac{\partial D}{\partial A_i})^T$; it follows from the tensor product representation (5) that if all the matrices A_i are hermitian (i.e. $A_i = A_i^*$) then the mixed discriminant $D(A_1, \dots, A_n)$ is a real number and $Q_i = Q_i^*$ also ($1 \leq i \leq n$).

All previous facts 1 – ... – 7 are well known (see, e.g., [3]). The next, Fact 8, is quite simple, but seems to be unknown (at least to the author) .

Fact 8.(Eulerian matrix identity)

$$\sum_{i=1}^n \langle Q_i \omega, A_i^* \omega \rangle = \sum_{i=1}^n \langle A_i Q_i \omega, \omega \rangle = D(A_1, \dots, A_n) \cdot \langle \omega, \omega \rangle, \quad (10)$$

where $\langle \omega, \omega \rangle$ stands for the standard inner product in C^n .

Proof: Consider the following identity :

$$D(XA_1, \dots, XA_i, \dots, XA_n) = \det(X) \cdot D(A_1, \dots, A_n), \quad (11)$$

where X is real symmetric nonsingular matrix . Differentiate its left and right sides respect to this matrix X :

$$\sum_{i=1}^n \hat{Q}_i^T A_i^T = \det(X) X^{-1} \cdot D(A_1, \dots, A_n). \quad (12)$$

Here $\hat{Q}_i^T = \frac{\partial D}{\partial \bullet_i}$ evaluated at $(XA_1, \dots, XA_i, \dots, XA_n)$.

Putting $X = I$ we get that

$$\sum_{i=1}^n Q_i^T A_i^T = D(A_1, \dots, A_n) \cdot I = \sum_{i=1}^n A_i Q_i \quad (13)$$

■

3 Basic Facts About Minimizers

3.1 Indecomposability

Following [4],[5] we call n -tuple $A = (A_1, A_2, \dots, A_n)$ consisting of positive semidefinite hermitian matrices indecomposable if $\text{Rank}(\sum_{i=1}^k A_{i_j}) > k$ for all $1 \leq i_1 < i_2 < \dots < i_k \leq n$, where $1 \leq k < n$.

Also, as in [4],[5] associate with a given n -tuple $\mathbf{A} = (A_1, \dots, A_n)$ a set $M(\mathbf{A})$ of n by n matrices $M(\mathbf{A}, W)$, where the indices W run over the orthogonal group $O(n)$, i.e. $WW^* = W^*W = I$, and for any orthogonal matrix W with columns $\omega_1, \dots, \omega_n$ the matrix $M(\mathbf{A}, W)$ has as its (i, j) entry the inner product $\langle A_j \omega_i, \omega_i \rangle$. Matrices $M(\mathbf{A}, W)$ inherit many properties of \mathbf{A} , which means that sometimes we can simplify things by dealing with matrices and not n -tuples.

Let us write down a few of these shared properties in the following proposition.

Proposition 3.1:

1. *The n -tuple \mathbf{A} is nonnegative (i.e. consists of positive semidefinite hermitian matrices) iff for all $W \in O(n)$, the matrix $M(\mathbf{A}, W)$ has nonnegative entries.*
2. *The n -tuple \mathbf{A} is indecomposable iff for all $W \in O(n)$, the matrix $M(\mathbf{A}, W)$ is fully indecomposable in the sense of [17].*
3. *$\sum_{i=1}^n a_i = I$ iff for all $W \in O(n)$, the matrix $M(\mathbf{A}, W)$ is row stochastic.*
4. *$\text{tr}(A_i) = 1$, for $1 \leq i \leq n$, iff for all $W \in O(n)$, the matrix $M(\mathbf{A}, W)$ is column stochastic.*
5. *Therefore \mathbf{A} is doubly stochastic iff for all $W \in O(n)$, the matrix $M(\mathbf{A}, W)$ is doubly stochastic.*

The following fact [4],[5] states that doubly stochastic n -tuples can be decomposed into indecomposable doubly stochastic tuples.

Fact 9.

Let $\mathbf{A} = (A_1, \dots, A_n)$ be a doubly stochastic n -tuple. Then there exists a partition $C_1 \cup \dots \cup C_k$ of $\{1, \dots, n\}$ such that

1. For all $1 \leq s \leq k$

$$\dim(\text{Im}(\sum_{i \in C_s} A_i)) =: \dim X_s = |C_s| =: c_s.$$

2. The linear subspaces X_s ($1 \leq s \leq k$) are pairwise orthogonal and $C^n = X_1 \oplus X_2 \oplus \dots \oplus X_k$.

It follows from the definition that, in the notation of Fact 9, the following identity holds :

$$D(\mathbf{A}) = D(\mathbf{A}_1) \cdots D(\mathbf{A}_s) \cdots D(\mathbf{A}_k),$$

where \mathbf{A}_s is a doubly stochastic c_s -tuple formed of restrictions of matrices $A_i (i \in C_s)$ on the subspace $X_s = \text{Im}(\sum_{i \in C_s} A_i)$.

3.2 Fritz John's Optimality Conditions

Recall that we are to find the minimum of $D(\mathbf{A})$ on D_n . The set of doubly stochastic n -tuples is characterized by the following constraints

$$\sum_{i=1}^n A_i = I, \text{tr} A_i = 1, A_i \succeq 0.$$

Notice that our constrained optimization problem is defined on the linear space $H =: H_n \oplus \cdots \oplus H_n$, where H_n is the linear space of $n \times n$ hermitian matrices. We view the linear space H_n as a n^2 dimensional real linear space with the inner product $\langle A, B \rangle = \text{tr}(AB)$. This inner product extends to tuples via standard summation. Though a rather straightforward application of John's Theorem [7] (see [4]) gives the next result, we decided to include a proof to make this paper self-contained.

Definition 3.2: Consider a doubly stochastic n -tuple $\mathbf{A} = (A_1, A_2, \dots, A_n)$. Present positive semidefinite matrices $A_i \succeq 0$ in the following block form with respect to the orthogonal decomposition $C^n = \text{Im}(A_i) \oplus \text{Ker}(A_i)$:

$$A_i = \begin{pmatrix} \tilde{A}_i & 0 \\ 0 & 0 \end{pmatrix}, A_i \succ 0; 1 \leq i \leq n. \quad (14)$$

Define a cone of admissible directions as follows:

$$K_0 = \{(Z_1, Z_2, \dots, Z_n) : \text{there exists } \epsilon > 0 \text{ such that the tuple}$$

$$(A_1 + \epsilon Z_1, A_2 + \epsilon Z_2, \dots, A_n + \epsilon Z_n) \text{ is doubly stochastic.}\}$$

I.e. K_0 is a minimal convex cone in the linear space of hermitian n -tuples $H =: H_n \oplus \cdots \oplus H_n$, which contains all n -tuples $\{\mathbf{B} - \mathbf{A} : \mathbf{B} \in D_n\}$. We also define the following two convex cones K_1, K_2 and one linear subspace K_3 of $H =: H_n \oplus \cdots \oplus H_n$:

$$K_1 = \{(B_1, B_2, \dots, B_n), \text{ where the matrices } B_i \text{ are hermitian and}$$

$$B_i = \begin{pmatrix} B_{i;1,1} & B_{i;1,2} \\ B_{i;2,1} & B_{i;2,2} \end{pmatrix}; \text{Im}(B_{i;2,1}) \subset \text{Im}(B_{i;2,2}), B_{i;2,2} \succeq 0, 1 \leq i \leq n.$$

$K_2 = \{(B_1, B_2, \dots, B_n), \text{ where the matrices } B_i \text{ are hermitian and}$

$$B_i = \begin{pmatrix} B_{i;1,1} & B_{i;1,2} \\ B_{i;2,1} & B_{i;2,2} \end{pmatrix}; B_{i;2,2} \succeq 0, 1 \leq i \leq n.$$

$$K_3 = \{(C_1, C_2, \dots, C_n) : C_i \in H_n, \text{ tr}(C_i) = 0 (1 \leq i \leq n) \text{ and } C_1 + \dots + C_n = 0\}.$$

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Proposition 3.3:

1. $K_0 = K_1 \cap K_3$.
2. The closure $\overline{K_1} = K_2$.
3. The closure $\overline{K_0} = K_2 \cap K_3$.

Proof:

1. Recall that a hermitian block matrix with strictly positive definite block $D_{1,1} \succ 0$

$$D = \begin{pmatrix} D_{1,1} & D_{1,2} \\ D_{2,1} & D_{2,2} \end{pmatrix}$$

is positive semidefinite iff $D_{2,2} \succeq 0$ and $D_{2,2} \succeq D_{2,1} D_{1,1}^{-1} D_{2,1}^*$. This proves that an n -tuple $(B_1, B_2, \dots, B_n) \in K_1$ iff there exists $\epsilon > 0$ such that $A_i + \epsilon B_i \succeq 0, 1 \leq i \leq n$. Intersection with K_3 just enforces the linear constraints $\text{tr}(A_i) = 1, 1 \leq i \leq n$ and $\sum_{1 \leq i \leq n} A_i = I$.

2. This item is obvious.
3. Clearly, the closure $\overline{K_1 \cap K_3} \subset \bar{K}_1 \cap \bar{K}_3 = K_2 \cap K_3$. We need to prove the reverse inclusion $\bar{K}_1 \cap \bar{K}_3 = K_2 \cap K_3 \subset \overline{K_1 \cap K_3}$. Consider the following hermitian n -tuple

$$\Delta_i = \frac{1}{n}I - A_i, \Delta_i = \begin{pmatrix} \frac{1}{n}I - \tilde{A}_i & 0 \\ 0 & \frac{1}{n}I \end{pmatrix}; 1 \leq i \leq n.$$

Clearly, the tuple $(\Delta_1, \dots, \Delta_n)$ is admissible. The important thing is that the $(2, 2)$ blocks of matrices Δ_i are strictly positive definite. Therefore if $(B_1, \dots, B_n) \in K_2 \cap K_3$ then for all $\epsilon > 0$ the tuple $(B_1 + \epsilon \Delta_1, \dots, B_n + \epsilon \Delta_n) \in K_0 = K_1 \cap K_3$. This proves that $\bar{K}_1 \cap \bar{K}_3 = K_2 \cap K_3 \subset \overline{K_1 \cap K_3}$ and thus that $\bar{K}_0 = K_2 \cap K_3$.

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Theorem 3.4: *If a doubly stochastic n -tuple $\mathbf{A} = (A_1, A_2, \dots, A_n)$ is a (local) minimizer then there exists a hermitian matrix R and scalars $\mu_i (1 \leq i \leq n)$ such that*

$$\left(\frac{\partial D}{\partial A_i}\right)^T =: Q_i = R + \mu_i I + P_i, \quad (15)$$

where the matrices P_i are positive semi-definite and $A_i P_i = P_i A_i = 0, 1 \leq i \leq n$.

Proof:

If a doubly stochastic n -tuple \mathbf{A} is a (local) minimizer then $\langle Q_1, Z_1 \rangle + \dots + \langle Q_n, Z_n \rangle \geq 0$ for all admissible tuples $(Z_1, Z_2, \dots, Z_n) \in K_0$. In other words the hermitian n -tuple (Q_1, Q_2, \dots, Q_n) belongs to a dual cone K'_0 . It is well known and obvious that a dual cone \bar{K}'_0 of a closure is equal to K'_0 . By Proposition 3.3 the closed cone $\bar{K}_0 = K_2 \cap K_3$, and the convex cones K_2, K_3 are closed. Therefore, see, for instance, [18], the closed convex dual cone $\bar{K}'_0 = K'_2 + K'_3$. We get by a straightforward inspection that

$$K'_2 = \{(P_1, P_2, \dots, P_n) : P_i = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{P}_i \end{pmatrix}, \tilde{P}_i \succeq 0, 1 \leq i \leq n,$$

$K'_3 = \{(R + \mu_1, R + \mu_2, \dots, R + \mu_n), \text{ where the matrix } R \text{ is hermitian and } \mu_i, 1 \leq i \leq n \text{ are real. Therefore, we get that } Q_i^T = R + \mu_i I + P_i, \text{ for some hermitian matrix } R, \text{ real } \mu_i \text{ and positive semidefinite } P_i \succeq 0 \text{ satisfying the equality } A_i P_i = P_i A_i = 0, 1 \leq i \leq n. \blacksquare\}$

Corollary 3.5: *In notations of Theorem 3.4 the following identities hold:*

$$D(\mathbf{A}) = \text{tr}(A_i \cdot Q_i) = \text{tr}(A_i(R + \mu_i I);$$

$$\langle A_i \omega, (R + \mu_i I) \omega \rangle = \langle A_i \omega, Q_i \omega \rangle, \omega \in C^n;$$

$$D(\mathbf{A}) = \sum_{1 \leq i \leq n} \langle A_i \omega, (R + \mu_i I) \omega \rangle, \langle \omega, \omega \rangle = 1.$$

Proof: It follows directly from the identity $A_i P_i = 0$, Facts 7,8 and the hermiticity of all the matrices involved here. \blacksquare

The following simple Lemma will be used in the proof of uniqueness.

Lemma 3.6: *Let us consider a doubly stochastic n -tuple*

$$\mathbf{A} = (A_1, A_2, \frac{1}{n}I \dots, \frac{1}{n}I),$$

where $A_1, A_2 \geq 0, \text{tr} A_1 = \text{tr} A_2 = 1$ and $A_1 + A_2 = \frac{2}{n}I$. Then $D(\mathbf{A}) = \frac{n!}{n^n} + \text{tr}((A_1 - \frac{1}{n}I) \cdot (A_1 - \frac{1}{n}I)^*) \frac{(n-2)!}{n^{n-2}}$.

Proof: First, notice that the matrices in this tuple commute. Thus, $D(\mathbf{A}) = \text{per}(C_{ij}(1 \leq i, j \leq n))$, where the first column of matrix C is equal to $\frac{1}{n}e + \tau$, second column to $\frac{1}{n}e - \tau$; all other columns are equal to $\frac{1}{n}e$. Here, as usual, e stands for vector of all ones; the vector τ consists of eigenvalues of $A_1 - \frac{1}{n}I$. Notice that $\sum \tau_i = 0$. Using the linearity of the permanent in each column we get that

$$\text{per}(\alpha_{ij}) = \frac{n!}{n^n} + \text{Per}(B),$$

where the first and second columns of matrix B are equal to τ and all others to $\frac{1}{n}e$.

An easy computation gives that

$$\text{Per} B = -2\left(\sum_{i < j} \tau_i \tau_j\right) \frac{(n-2)!}{n^{n-2}}.$$

But $0 = (\tau_i + \dots + \tau_n)^2 = \tau_1^2 + \dots + \tau_n^2 + 2 \sum_{i < j} \tau_i \tau_j$. Thus

$$D(\mathbf{A}) = \text{per}(C) = \frac{n!}{n^n} + \frac{(n-2)!}{n^{n-2}} \cdot \left(\sum_{i=1}^n \tau_i^2\right) = \frac{n!}{n^n} + \frac{(n-2)!}{n^{n-2}} \text{tr}\left((A_i - \frac{1}{n}I)(A_i - \frac{1}{n}I)^*\right).$$

■

4 Proof of Bapat's conjecture

Theorem 4.1:

1. $\min_{\mathbf{A} \in D_n} D(\mathbf{A}) = \frac{n!}{n^n}$.
2. The minimum is uniquely attained at $J_n = (\frac{1}{n}I, \frac{1}{n}I, \dots, \frac{1}{n}I)$.

Proof:

1. To make our proof a bit simpler, we will prove the first part of Theorem 4.1 by induction. Assume that the theorem is true for $m < n$. (Case $n = 1$ is obvious). Therefore, had a minimizing tuple \mathbf{A} decomposed into two tuples B_1 and B_2 , of dimensions m_1 and m_2 respectively, this would imply, using Fact 9, that $D(A) = D(B_1)D(B_2) \geq \frac{m_1!}{m_1} \frac{m_2!}{m_2} > \frac{n!}{n^n}$.

The last inequality is clearly wrong as $D(A) \leq D(J_n) = \frac{n!}{n^n}$. Thus we can assume that any minimizing tuple is fully indecomposable. Now apply to a minimizing tuple $\mathbf{A} = (A_1, \dots, A_n)$ Theorem 3.4: there exists a Hermitian matrix R and scalars μ_1, \dots, μ_n such that $P_i = Q_i - R - \mu_i I \geq 0$ and $A_i P_i = 0$. From Corollary 3.5 we get that $D(A) = \text{tr}(A_i(R + \mu_i I))$ and $D(A) = \sum_{i=1}^n \langle A_i \omega, (R + \mu_i I) \omega \rangle$ for any normed vector $\omega \in C^n$.

Let $W^* R W = \text{Diag}(\theta_1, \dots, \theta_n)$ for some real θ_i and unitary W . As in Proposition 3.1, define a doubly stochastic matrix $B = M(A, W)$. i.e. $b_{ij} = \langle A_i \omega_j, \omega_j \rangle$. Here ω_j is a j th column of W (or j th eigenvector of R). Writing identities $D(A) = \text{tr}(A_i(R + \mu_i I))$

and $\sum_{i=1}^n < A_i \omega_j, (R + \mu_i I) \omega_j > = D(A)(1 \leq j \leq n)$ in terms of the matrix B , we obtain the following systems of linear equations

$$\mu_i + \sum_{j=1}^n b_{ij} \theta_j = D(A)(1 \leq i \leq n);$$

$$\theta_j + \sum_{i=1}^n B_{ij} \mu_i = D(A)(1 \leq j \leq n).$$

These equations are, of course, encountered also in the matrix case, where they led to the crucial London's Lemma [11], [17], [12], [14]. Proceeding in exactly the same way, we easily deduce that $\mu = (\mu_1, \dots, \mu_n)$ is an eigenvector of BB^T with eigenvalue 1; θ is an eigenvector of $B^T B$ with eigenvalue 1.

It follows from Proposition 3.1 that B is fully indecomposable, which is equivalent to the fact that 1 is a simple eigenvalue for both BB^T and $B^T B$ (see e.g. [17]). Therefore both μ and θ are proportional to the vector e (all ones). Thus $\mu_i = \alpha$, $\theta_i = \beta$ and $\alpha + \beta = D(\mathbf{A})$. Returning to matrices $R + \mu_i I$ we get that $R + \mu_i I = D(\mathbf{A}) \cdot I$.

Now comes the punch line, a generalization of London's Lemma [11] to mixed discriminants:

For a minimizing tuple $\mathbf{A} = (A_1 \cdots A_n)$ the following inequality holds:

$$\left(\frac{D(A)}{\partial A_i}\right)^T =: Q_i \succeq R + \mu_i I = D(\mathbf{A}) \cdot I. \quad (16)$$

Indeed, by Theorem 3.4 $Q_i = R + \mu_i I + P_i$ and $P_i \succeq 0$. After the inequality (16) is established, we are back to familiar grounds of the Van der Waerden's conjecture proofs [12], [14]. Let us introduce the following notations:

$$\mathbf{A}^{i,j} = (A_1, \dots, A_i, \dots, A_i, \dots, A_n), f_{i,j}(\mathbf{A}) = \frac{\mathbf{A}^{i,j} + \mathbf{A}^{j,i}}{2}.$$

I.e., we replace j th matrix in the tuple by the i th one to define $\mathbf{A}^{i,j}$, replace i th and j th matrices by their arithmetic average to define $f_{i,j}(\mathbf{A})$.

The celebrated Alexandrov-Fenchel inequalities state that

1. $D(\mathbf{A})^2 \geq D(\mathbf{A}^{i,j}) \cdot D(\mathbf{A}^{j,i})$.
2. If $A_i \succ 0 (1 \leq i \leq n)$ and $D(\mathbf{A})^2 = D(\mathbf{A}^{i,j}) \cdot D(\mathbf{A}^{j,i})$ then $A_i = \tau A_j$ for some positive τ .

Suppose that \mathbf{A} is a minimal doubly stochastic n -tuple, i.e. that $D(\mathbf{A}) = \min_{\mathbf{B} \in D_n} D(\mathbf{B})$. Then for $i \neq j$, $f_{i,j}(\mathbf{A})$ is also minimal doubly stochastic n -tuple. Indeed, double stochasticity is obvious.

By the linearity of mixed discriminants in each matrix argument we get that $D(f_{i,j}(\mathbf{A})) = \frac{1}{2}D(\mathbf{A}) + \frac{1}{4}D(\mathbf{A}^{i,j}) + \frac{1}{4}D(\mathbf{A}^{j,i})$. Also,

$$D(\mathbf{A}^{i,j}) = \text{tr}(A_i \cdot Q_j), D(\mathbf{A}^{j,i}) = \text{tr}(A_j \cdot Q_i). \quad (17)$$

(We used here Fact 7).

As $Q_k \succeq D(A) \cdot I$ ($1 \leq k \leq n$) from inequality (16) and $\text{tr} A_k \equiv 1$ then $D(\mathbf{A}^{i,j}) \geq D(A)$ and $D(A^{j,i}) \geq D(A)$. Thus, we conclude based on Alexandrov-Fenchel's inequalities that if \mathbf{A} is a minimal doubly stochastic n -tuple then $D(\mathbf{A}^{i,j}) \equiv D(\mathbf{A})$ ($i \neq j$) and $D(f_{i,j}(A)) = D(\mathbf{A})$.

Additionally, if a minimal tuple $\mathbf{A} = (A_1, \dots, A_n)$ consists of positive definite matrices then $A_i = A_j$ ($i \neq j$). Indeed $A_i = \tau A_j$ and $\text{tr} A_i = \text{tr} A_j = 1$, thus $\tau = 1$. As $A_1 + \dots + A_n = I$, we conclude that the only "positive" minimal doubly stochastic n -tuple is $J_n = (\frac{1}{n}I, \dots, \frac{1}{n}I)$. In any case, as $D(\mathbf{A}^{i,j}) \equiv D(\mathbf{A})$ ($i \neq j$), then $D(f_{i,j}(\mathbf{A})) = D(\mathbf{A})$ for minimal tuples \mathbf{A} .

Define the following iteration on n -tuples:

$$\begin{aligned} \mathbf{A}_0 &= \mathbf{A}, \\ \mathbf{A}_1 &= f_{1,2}(A_0) \\ &\dots \\ \mathbf{A}_{n-1} &= f_{n-1,n}(A_{n-2}) \\ \mathbf{A}_n &= f_{1,2}(A_{n-1}) \\ &\dots \end{aligned}$$

It is clear that

$$\mathbf{A}_k \rightarrow \left(\frac{A_1 + \dots + A_n}{n}, \dots, \frac{A_1 + \dots + A_n}{n} \right).$$

As the initial tuple satisfies $A_1 + \dots + A_n = I$, then $A_k \rightarrow (\frac{1}{n}I, \dots, \frac{1}{n}I)$. (Indeed $Pr_1 = f_{1,2}, \dots, Pr_{n-1} = f_{n-1,n}$ are orthogonal projectors in the linear finite-dimensional Hilbert space of hermitian n -tuples. By a well known result, $\lim_{k \rightarrow \infty} (Pr_{n-1} \dots Pr_1)^k = Pr$, where Pr is an orthogonal projector on the linear subspace $L = \text{Im}(Pr_1) \cap \text{Im}(Pr_2) \cap \dots \cap \text{Im}(Pr_{n-1})$. It is obvious that $L = \{(A_1, \dots, A_n) : A_1 = A_2 = \dots = A_n\}$.) As the mixed discriminant is a continuous map from tuples to reals, we conclude that $J_n = (\frac{1}{n}I, \dots, \frac{1}{n}I)$ is a minimal tuple and $\min_{\mathbf{A} \in D_n} D(\mathbf{A}) = D(J_n) = \frac{n!}{n^n}$.

2. Let us now prove the uniqueness. We only have to prove that any minimal doubly stochastic n -tuple consist of positive definite matrices. Suppose that not. Then there exists an integer $k \geq 0$ such that the tuple \mathbf{A}_k has at least one singular matrix and the next tuple \mathbf{A}_{k+1} consists of positive definite matrices.

Assume without loss of generality that $\mathbf{A}_{k+1} = f_{1,2}(\mathbf{A}_k)$ and $\mathbf{A}_k = (A_1, A_2, A_3, \dots, A_n)$.

Then $\frac{A_1 + A_2}{2} = A_3 = \dots = A_n = \frac{1}{n}I$. From Lemma 3.6 we conclude that

$$D(\mathbf{A}_k) = D(J_n) + \frac{(n-2)!}{n^{n-2}} \cdot \text{tr}((A_2 - \frac{1}{n}I) \cdot (A_2 - \frac{1}{n}I)^*).$$

As we already know that J_n is a minimal tuple and \mathbf{A}_k is also a minimal tuple, thus $A_1 = A_2 = \frac{1}{n}I$. But at least one of A_1, A_2 suppose to be singular. We got the desired contradiction.

■

Corollary 4.2: *Let us define the set $D_{n,P}$ of hermitian n -tuples as follows*

$$D_{n,P} = \{\mathbf{A} = (A_1, A_2, \dots, A_n) : A_i \text{ is positive semi-definite, } \text{tr}(A_i) \equiv 1 \text{ and } \sum_{i=1}^n A_i = P.$$

Notice that $\text{tr}(P) = n$ necessarily. If the matrix P is sufficiently close to the identity matrix I then $\min_{A \in D_{n,P}} D(A) = \frac{n!}{n^n} \det(P)$.

Proof: It follows from the uniqueness part of Theorem 4.1 that if P is sufficiently close to the identity matrix I then there exists a minimal n -tuple $\mathbf{A} = (A_1, A_2, \dots, A_n) \in D_{n,P}$ consisting of positive definite matrices. Very similarly to Theorem 3.4, it follows that $Q_i = R + \mu_i I$ ($1 \leq i \leq n$), μ_i is real and R is hermitian. It is straightforward to prove that under this condition $D(f_{i,j}(A)) = D(\mathbf{A})$. Also, $f_{i,j}(A) \in D_{n,P}$. Indeed,

$$D(\mathbf{A}^{i,j}) = \text{tr}(A_i \cdot Q_j) = \text{tr}(A_i \cdot (R + \mu_j I)) \text{ and } D(\mathbf{A}) = \text{tr}(A_i \cdot (R + \mu_i I)). \quad (18)$$

Thus, $D(\mathbf{A}^{i,j}) = \text{tr}(A_i)(\mu_j - \mu_i) + D(\mathbf{A}) = D(\mathbf{A}) + (\mu_j - \mu_i)$.
Therefore,

$$D(f_{i,j}(\mathbf{A})) = \frac{1}{2}D(\mathbf{A}) + \frac{1}{4}D(\mathbf{A}^{i,j}) + \frac{1}{4}D(\mathbf{A}^{j,i}) = D(\mathbf{A}).$$

As in the proof of Theorem 4.1, the last equality leads to the minimality of the tuple $(\frac{P}{n}, \dots, \frac{P}{n})$.

■

5 Motivations and Connections

The author came across Bapat's conjecture because of the following theorem [4],[5].

Theorem 5.1: *Let us consider an indecomposable n -tuple $\mathbf{A} = (A_1, \dots, A_n)$ consisting of positive semi-definite matrices. Then*

1. *There exist a unique vector α of positive scalars α_i ; $1 \leq i \leq n$ with product equal to 1 and a positive definite matrix S , such that the n -tuple $\mathbf{B} = (B_1, \dots, B_n)$, defined by $B_i = \alpha_i S A_i S$, is doubly stochastic.*

2. *The vector α above is the unique minimum of $\det(\sum t_i A_i)$ on the set of positive vectors with product equal to 1 and $\min_{x_i > 0, \prod_{i=1}^N x_i = 1} \det(\sum x_i A_i) = (\det(S))^{-2}$.*

Let us define the following important quantity, the capacity of \mathbf{A} :

$$Cap(\mathbf{A}) = \inf_{x_i > 0, \prod_{i=1}^N x_i = 1} \det(\sum x_i A_i).$$

Using Theorem 4.1 and Theorem 5.1 we get the following inequality [4], [5]:

$$1 \leq \frac{Cap(\mathbf{A})}{D(\mathbf{A})} \leq \frac{n^n}{n!}. \quad (19)$$

This last inequality played the most important role in [4], [5]. With many other technical details it led to a deterministic poly-time algorithm to approximate mixed volumes of ellipsoids within a simply exponential factor. In the following subsection we will use it to obtain a rather unusual extension of the Alexandrov-Fenchel inequality.

5.1 Generalized Alexandrov-Fenchel inequalities

Define L_n as a set of all integer vectors $\alpha = (\alpha_1, \dots, \alpha_n)$ such that $\alpha_i \geq 0$ and $\sum_{i=1}^n \alpha_i = n$.

For an integer vector

$$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) : \alpha_i \geq 0, \sum \alpha_i = N$$

we define a matrix tuple

$$\mathbf{A}^{(\alpha)} = (\underbrace{A_1, \dots, A_1}_{\alpha_1}, \dots, \underbrace{A_k, \dots, A_k}_{\alpha_k}, \dots, \underbrace{A_n, \dots, A_n}_{\alpha_n})$$

i.e., matrix A_i has α_i copies in $\mathbf{A}^{(\alpha)}$. For a vector $x = (x_1, \dots, x_n)$, we define a monomial $x^{(\alpha)} = x_1^{\alpha_1} \dots x_n^{\alpha_n}$.

We will use the notations $M^{(\alpha)}$ for the mixed discriminant $D(\mathbf{A}^{(\alpha)})$ and $Cap^{(\alpha)}$ for the capacity $Cap(\mathbf{A}^{(\alpha)})$.

Theorem 5.2: Consider a tuple $A = (A_1, \dots, A_n)$ of semidefinite hermitian $n \times n$ matrices. If vectors $\alpha, \alpha^1, \dots, \alpha^m$ belong to L_n and

$$\alpha = \sum_{i=1}^m \gamma_i \alpha^i; \quad \gamma_i \geq 0, \quad \sum \gamma_i = 1$$

then the following hold inequalities hold:

$$\log(Cap^{(\alpha)}) \geq \sum \gamma_i \log(Cap^{(\alpha^i)}), \quad (20)$$

$$\log(M^{(\alpha)}) \geq \sum \gamma_i \log(M^{(\alpha^i)}) - \log\left(\frac{n^n}{n!}\right). \quad (21)$$

Proof: Notice that the inequality (21) follows from (20) via a direct application of the inequality (19). It remains to prove (20).

First, using the arithmetic/geometric means inequality, we get that $Cap^{(\alpha)} \geq d$ iff

$$\log(\det(\sum_{i=1}^n A_i \alpha_i e^{x_i})) \geq \sum \alpha_i x_i + \log d$$

for all real vectors $x = (x_1, \dots, x_n)$.

Now, suppose that

$$\alpha, \alpha^1, \dots, \alpha^m \in L_n$$

and

$$\alpha = \sum_{i=1}^m \gamma_i \alpha^i, \quad \gamma_i \geq 0, \quad \sum_{i=1}^m \gamma_i = 1.$$

Then

$$\log(\det(\sum_{i=1}^n A_i \alpha_i^j e^{x_i})) \geq \langle \alpha^j, x \rangle + \log(Cap^{(\alpha^j)}).$$

Multiplying each of the inequalities above by the corresponding γ_i and adding afterwards we get that

$$\sum_{j=1}^m \gamma_j \log(\det(\sum_{i=1}^n A_i \alpha_i^j e^{x_i})) \geq \langle \alpha, X \rangle + \sum_{j=1}^m \gamma_j \log(Cap^{(\alpha^j)}).$$

As $\log(\det(X))$ is concave for $X \succ 0$, we eventually get the inequality

$$\log(Cap^{(\alpha)}) \geq \sum_{j=1}^m \gamma_j \log(Cap^{(\alpha^j)}).$$

■

The Alexandrov-Fenchel inequalities can be written as

$$\log(M^{(\alpha)}) \geq \frac{\log(M^{(\alpha^1)}) + \log(M^{(\alpha^2)})}{2}.$$

Here the vector $\alpha = (1, 1, \dots, 1)$; $\alpha^1 = (2, 0, 1, \dots, 1)$; $\alpha^2 = (0, 2, 1, \dots, 1)$. Perhaps the extra $-\log(\frac{n^n}{n!})$ in Theorem 5.2 is just an artifact of our proof? We will show below that an extra factor is needed indeed.

Define $AF(n)$ as the smallest (possibly infimum) constant one has to subtract from $\sum \gamma_i \log(M^{(\alpha^i)})$ in the right side of (21) in order to get the inequality. Then by Theorem 5.2, $AF(n) \leq \log(\frac{n^n}{n!}) \cong n$. We will prove below that $AF(n) \geq n \log(\sqrt{2})$ even for tuples consisting of diagonal matrices. In this diagonal case the mixed discriminant coincides with the permanent.

Consider the following $N \times N$ matrix with nonnegative entries:

$$B = \begin{pmatrix} 1 & 1 & \ddots & \ddots & 0 \\ 0 & 1 & \ddots & \ddots & \ddots \\ 0 & & \ddots & \ddots & \\ 1 & & & \ddots & 1 \end{pmatrix} \quad \text{i.e. } B = I + J$$

where J is a cyclic shift. Assume without a “big” loss of generality that $N = 2k$. Define

$$\alpha^1 = (\underbrace{2, 2, 2, \dots, 2}_k, 0, 0, \dots, 0), \quad \alpha^2 = (\underbrace{0, 0, 0, \dots, 0}_k, 2, 2, \dots, 2)$$

Then, $e = \frac{\alpha^1 + \alpha^2}{2}$, $e = (1, 1, \dots, 1)$ and $\text{per}(B^{(e)}) = 2$, $\text{per}(B^{(\alpha^1)}) = \text{per}(B^{(\alpha^2)}) = 2^k = 2^{\frac{N}{2}}$. (Here the notation $B^{(\alpha)}$ stands for the matrix having α_i copies of i th column of the matrix B .) Thus,

$$\frac{\text{per}(B^{(e)})}{\sqrt{\text{per}(B^{(\alpha^1)}) \cdot \text{per}(B^{(\alpha^2)})}} = \frac{2}{2^{\frac{N}{2}}} \cong \sqrt{2}^{-N}.$$

Conjecture 5.3:

$$\lim_{n \rightarrow \infty} \frac{AF(n)}{n} = 1.$$

■

6 Further analogs of van der Waerden conjecture

6.1 4-dimensional Pascal’s determinants

The following open question has been motivated by the author’s study of the quantum entanglement [8].

Consider a block matrix

$$\rho = \begin{pmatrix} A_{1,1} & A_{1,2} & \dots & A_{1,n} \\ A_{2,1} & A_{2,2} & \dots & A_{2,n} \\ \dots & \dots & \dots & \dots \\ A_{n,1} & A_{n,2} & \dots & A_{n,n} \end{pmatrix}, \quad (22)$$

where each block is a $n \times n$ complex matrix. Define

$$QP(\rho) =: \sum_{\sigma \in S_n} (-1)^{\text{sign}(\sigma)} D(A_{1,\sigma(1)}, \dots, A_{n,\sigma(n)}), \quad (23)$$

where $D(A_1, \dots, A_n)$ is the mixed discriminant. If instead of the block form (22), to present $\rho = \{\rho(i_1, i_2, i_3, i_4) : 1 \leq i_1, i_2, i_3, i_4 \leq n\}$, e.g. as a 4-dimensional tensor, then

$$\begin{aligned} QP(\rho) &= \frac{1}{N!} \sum_{\tau_1, \tau_2, \tau_3, \tau_4 \in S_N} (-1)^{\text{sign}(\tau_1 \tau_2 \tau_3 \tau_4)} \\ &\quad \prod_{i=1}^N \rho(\tau_1(i), \tau_2(i), \tau_3(i), \tau_4(i)). \end{aligned} \quad (24)$$

In other words, $QP(\rho)$ is, up to $\frac{1}{N!}$ factor, equal to the 4-dimensional Pascal’s determinant [16].

Call such a block matrix ρ doubly stochastic if the following conditions hold:

1. $\rho \succeq 0$, e.g. the $n^2 \times n^2$ matrix ρ is positive semidefinite.
2. $\sum_{1 \leq i \leq n} A_{i,i} = I$.
3. The matrix of traces $\{tr(A_{i,j}) : 1 \leq i, j \leq n\} = I$.

A positive semidefinite block matrix ρ is called separable if $\rho = \sum_{1 \leq i \leq k < \infty} P_i \otimes Q_i$, where the matrices $P_i \succeq 0, Q_i \succeq 0 : 1 \leq i \leq k$; nonseparable positive semidefinite block matrices are called entangled. Notice that in the block-diagonal case $QP(\rho) = D(A_{1,1}, \dots, A_{n,n})$ and our definition of double stochasticity coincides with double stochasticity of n -tuples from [3]. Let us denote the closed convex set of doubly stochastic $n \times n$ block matrices as BLD_n , a closed convex set of separable doubly stochastic $n \times n$ block matrices as SeD_n . It was shown in [8] that $\min_{\rho \in BLD_n} QP(\rho) = 0$ for $n \geq 3$ and, on the other hand, $\min_{\rho \in SeD_n} QP(\rho) > 0$ for $n \geq 1$.

Conjecture 6.1: $\min_{\rho \in SeD_n} QP(\rho) = \frac{n!}{n^n}$. ■

It is easy to prove this conjecture for $n = 2$, moreover the following equalities hold :

$$\min_{\rho \in BLD_2} QP(\rho) = \min_{\rho \in SeD_2} QP(\rho) = \frac{2!}{2^2} = \frac{1}{2}.$$

6.2 Hyperbolic polynomials

The following concept of hyperbolic polynomials originated in the theory of partial differential equations [1].

Definition 6.2: A homogeneous polynomial $p(x), x \in R^m$ of degree n in m real variables is called hyperbolic in the direction $e \in R^m$ (or e -hyperbolic) if for any $x \in R^m$ the polynomial $p(x - \lambda e)$ in the one variable λ has exactly n real roots counting their multiplicities. We assume in this paper that $p(e) > 0$.

Denote the ordered vector of roots of $p(x - \lambda e)$ as $\lambda(x) = (\lambda_1(x) \geq \lambda_2(x) \geq \dots \geq \lambda_n(x))$. It is well known that the product of roots is equal to $p(x)$. Call $x \in R^m$ e -positive (e -nonnegative) if $\lambda_n(x) > 0$ ($\lambda_n(x) \geq 0$). Define $tr_e(x) = \sum_{1 \leq i \leq n} \lambda_i(x)$.

A k -tuple of vectors (x_1, \dots, x_k) is called e -positive (e -nonnegative) if $x_i, 1 \leq i \leq k$ are e -positive (e -nonnegative). Let us fix n real vectors $x_i \in R^m, 1 \leq i \leq n$ and define the following homogeneous polynomial:

$$P_{x_1, \dots, x_n}(\alpha_1, \dots, \alpha_n) = p\left(\sum_{1 \leq i \leq n} \alpha_i x_i\right) \quad (25)$$

Following [2], we define the p -mixed value of an n -vector tuple $\mathbf{X} = (x_1, \dots, x_n)$ as

$$M_p(\mathbf{X}) =: M_p(x_1, \dots, x_n) = \frac{\partial^n}{\partial \alpha_1 \dots \partial \alpha_n} p\left(\sum_{1 \leq i \leq n} \alpha_i x_i\right) \quad (26)$$

Finally, call an n -tuple of real m -dimensional vectors (x_1, \dots, x_n) e -doubly stochastic if it is e -nonnegative. $tr_e x_i = 1 (1 \leq i \leq n)$ and $\sum_{1 \leq i \leq n} x_i = e$. Denote the closed convex set of e -doubly stochastic n -tuples as $HD_{e,n}$. ■

Example 6.3: Consider the following homogeneous polynomial $p(\alpha_1, \dots, \alpha_n) = \det(\sum_{1 \leq i \leq n} \alpha_i A_i)$. If $A_i \succeq 0 : 1 \leq i \leq n$ and $\sum_{1 \leq i \leq n} A_i \succ 0$ then $p(\cdot)$ is hyperbolic in the direction e , where e is a vector of all ones. If $\sum_{1 \leq i \leq n} A_i = I$ then e -double stochasticity of n -tuple $\mathbf{X} = (e_1, e_2, \dots, e_n)$ of n -dimensional canonical axis vectors is the same as double stochasticity of n -tuple of matrices $\mathbf{A} = (A_1, \dots, A_n)$. Moreover $M_p(\mathbf{X}) = D(\mathbf{A})$. ■

It was proved in a very recent paper [19] that $\min_{\mathbf{X} \in HD_{e,n}} M_p(\mathbf{X}) > 0$.

Conjecture 6.4: $\min_{\mathbf{X} \in HD_{e,n}} M_p(\mathbf{X}) = p(e) \frac{n!}{n^n}$ ■

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References

- [1] L. Garding, An inequality for hyperbolic polynomials, *Jour. of Math. and Mech.*, 8(6): 957-965, 1959.
- [2] A.G. Khovanskii, Analogues of the Aleksandrov-Fenchel inequalities for hyperbolic forms, *Soviet Math. Dokl.* 29(1984), 710-713.
- [3] R. Bapat, Mixed discriminants of positive semidefinite matrices, *Linear Algebra and its Applications* 126, 107-124, 1989.
- [4] L. Gurvits and A. Samorodnitsky, A deterministic polynomial-time algorithm for approximating mixed discriminant and mixed volume, *Proc. 32 ACM Symp. on Theory of Computing*, ACM, New York, 2000.
- [5] L. Gurvits and A. Samorodnitsky, A deterministic algorithm approximating the mixed discriminant and mixed volume, and a combinatorial corollary, *Discrete Comput. Geom.* 27: 531 -550, 2002.
- [6] S. Friedland, A lower bound for the permanent of a doubly stochastic matrix, *Annals of Mathematics*, 110(1979), 167-176.
- [7] F. John, Extremum problems with inequalities as subsidiary conditions, **Studies and Essays, presented to R. Courant on his 60th birthday**, Interscience, New York, 1948.
- [8] L. Gurvits, Classical deterministic complexity of Edmonds' problem and Quantum Entanglement, *Proc. 35 ACM Symp. on Theory of Computing*, ACM, New York, 2003.

- [9] A. Aleksandrov, On the theory of mixed volumes of convex bodies, IV, Mixed discriminants and mixed volumes (in Russian), *Mat. Sb. (N.S.)* 3 (1938), 227-251.
- [10] R. Schneider, **Convex bodies: The Brunn-Minkowski Theory**, Encyclopedia of Mathematics and Its Applications, vol. 44, Cambridge University Press, New York, 1993.
- [11] D. London, Some notes on the van der Waerden conjecture, *Linear Algebra and Appl.* 4 (1971), 155-160.
- [12] D. I. Falikman, Proof of the van der Waerden's conjecture on the permanent of a doubly stochastic matrix, *Mat. Zametki* 29, 6: 931-938, 957, 1981, (in Russian).
- [13] M. Grötschel, L. Lovasz and A. Schrijver, **Geometric Algorithms and Combinatorial Optimization**, Springer-Verlag, Berlin, 1988.
- [14] G.P. Egorychev, The solution of van der Waerden's problem for permanents, *Advances in Math.*, 42, 299-305, 1981.
- [15] A. Panov, On mixed discriminants connected with positive semidefinite quadratic forms, *Soviet Math. Dokl.* 31 (1985).
- [16] E. Pascal, *Die Determinanten*, Teubner-Verlag, Leipzig, 1900.
- [17] H. Minc, *Permanents*, Addison - Wesley, Reading, MA, 1978.
- [18] R. Tyrrell Rockafellar, *Convex analysis*, Princeton University Press, 1970.
- [19] L. Gurvits, Combinatorial and algorithmic aspects of hyperbolic polynomials, 2003 ; available at <http://xxx.lanl.gov/abs/math.CO/0404474>.